

Projective synchronization in fractional order chaotic systems and its control*

Chunguang Li[†]

Centre for Nonlinear and Complex Systems, School of Electronic Engineering,
University of Electronic Science and Technology of China, Chengdu, 610054, P. R. China.

Abstract

The chaotic dynamics of fractional (non-integer) order systems have begun to attract much attention in recent years. In this paper, we study the projective synchronization in two coupled fractional order chaotic oscillators. It is shown that projective synchronization can also exist in coupled fractional order chaotic systems. A simple feedback control method for controlling the scaling factor onto a desired value is also presented.

PACS. 05.45.Xt- Synchronization; coupled oscillators.

Although fractional calculus has a mathematical history nearly as long as that of the integer-order calculus, the applications of it to physics and engineering are just a recent focus of interest [1, 2]. Many systems are known to display fractional order dynamics, such as viscoelastic systems [3-5], dielectric polarization [6], electrode-electrolyte polarization [7] and electromagnetic waves [8], so it is important to study the properties of fractional order systems. The dynamics of fractional order systems has not yet been fully studied, and it is by no mean trivial. The definitions of fractional order calculus are geometrically and physically less intuitive than the integer-order ones, and can't be simulated directly in time-domain. It is still unclear whether the dynamics of fractional order systems is similar to the integer-order ones. More recently, many authors have begun to investigate the chaotic dynamics of fractional order dynamical systems [9-17]. In [9], it was shown that the fractional order Chua's system of order as low as 2.7 can produce a chaotic attractor. In [10], it was shown that nonautonomous Duffing systems of order less than 2 can still behave in a chaotic manner. In [11], chaotic behaviors of the fractional order "jerk" model was studied, in which chaotic attractor was obtained with system orders as low as 2.1, and in [12] the control of this fractional order chaotic system was reported. In [13], chaotic behavior of the fractional order Lorenz system was studied, but unfortunately, the results presented in this paper are not correct. In [14] and [15], bifurcation and chaotic dynamics of the fractional order cellular neural networks were studied. In [16], chaos and hyperchaos in the fractional order Rössler equations were studied, in which we showed that chaos can exist in the fractional order Rössler equation with

*Progress of Theoretical Physics, Vol. 115, No. 3, March 2006, pp.661-666.

[†]Email: cgli@uestc.edu.cn

order as low as 2.4, and hyperchaos can exist in the fractional order Rössler hyperchaos equation with order as low as 3.8. In [17], we studied the chaotic behavior and its control in the fractional order Chen system. In [18], the author presents a broad review of existing models of fractional kinetics and their connection to dynamical models, phase space topology, and other characteristics of chaos.

On the other hand, synchronization of chaotic systems has attracted much attentions [19] since the seminal paper by Pecora and Carroll [20]. Besides the identical (complete) synchronization of chaotic systems, some other types of synchronization are also very interesting cooperative behaviors of chaotic systems, among which are the phase synchronization [21], lag synchronization [22] and projective synchronization [23] of chaotic oscillators.

In [24], we have studied the synchronization of fractional order chaotic systems. And since then, the synchronization of fractional order chaotic systems has begun to attract attentions of some researchers, see for example [25]. But, in the literature, the authors are all concerned with the identical synchronization of fractional order chaotic systems. In [26], we studied the phase and lag synchronization of coupled fractional order chaotic oscillators. However, to our knowledge, projective synchronization in fractional order chaotic systems has not been discussed yet. In this paper, we address this topic.

Projective synchronization is the dynamical behavior in which the response of two identical systems synchronize up to a constant scaling factor. This phenomenon was observed in the coupled integer-order partially linear systems

$$\begin{aligned}\dot{\mathbf{u}}_m &= \mathbf{M}(z) \cdot \mathbf{u}_m, \\ \dot{z} &= f(\mathbf{u}_m, z), \\ \dot{\mathbf{u}}_s &= \mathbf{M}(z) \cdot \mathbf{u}_s\end{aligned}\tag{1}$$

where the matrix $\mathbf{M}(z)$ is only dependent on z , \mathbf{u}_m and \mathbf{u}_s are the master and slave state vectors, respectively. A partially linear system is a set of ordinary differential equations, whose state vector can be divided into two parts (\mathbf{u}, z) in such a way. The two partially linear systems are coupled through z : the z in the slave system will be the z of the master system. The above coupled system is said to be projective synchronous if for an initial condition there is a constant β such that asymptotically in time

$$\|\beta \mathbf{u}_m - \mathbf{u}_s\| \rightarrow 0.\tag{2}$$

In this paper, we study the projective synchronization in coupled fractional order partially linear chaotic systems of the form

$$\begin{aligned}\frac{d^\alpha \mathbf{u}_m}{dt^\alpha} &= \mathbf{M}(z) \cdot \mathbf{u}_m, \\ \frac{d^\alpha z}{dt^\alpha} &= f(\mathbf{u}_m, z), \\ \frac{d^\alpha \mathbf{u}_s}{dt^\alpha} &= \mathbf{M}(z) \cdot \mathbf{u}_s\end{aligned}\tag{3}$$

where α is the fractional order.

In the literature, there are several different definitions of fractional derivatives, see e.g. [1]. Perhaps the best known one is the Riemann-Liouville definition:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,\tag{4}$$

where $\Gamma(\cdot)$ is the gamma function and $n - 1 \leq \alpha < n$. The geometric and physical interpretation of the fractional derivatives were given in [27]. Upon considering the initial values to be zero, the Laplace transform of the Riemann-Liouville fractional derivative is $L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\}(s) = s^\alpha L\{f(t)\}$. So, the fractional integral operator of order “ α ” can be represented by the transfer function $F(s) = \frac{1}{s^\alpha}$.

According to the standard definition of the fractional differintegral, we can't directly implement the fractional operators in time-domain simulations. An efficient method to circumvent this problem is to approximate (in the frequency domain) the fractional operators by using the standard integer order operators. In the following simulations, we will use the approximation method proposed in [28], which was also adopted in [9, 11, 12, 14, 15, 16, 17]. In Table 1 of [9], the authors presented the approximations for $1/s^q$ with $q = 0.1 - 0.9$ in steps 0.1 with errors of approximately 2dB. We will use these approximations in our following simulations.

An often studied example of the partially linear chaotic system in projective synchronization in the integer-order case is the Lorenz system, but, unfortunately the results about the fractional order Lorenz system is not correct [13], so we cannot use this system as the example of fractional order partially linear chaotic system in our study. Here we use the fractional order Chen system [17],

$$\begin{aligned}\frac{d^\alpha x}{dt^\alpha} &= a(y - x), \\ \frac{d^\alpha y}{dt^\alpha} &= (c - a)x - xz + cy, \\ \frac{d^\alpha z}{dt^\alpha} &= xy - bz\end{aligned}\tag{5}$$

which is partially linear with $\mathbf{u} = (x, y)$ and

$$\mathbf{M}(z) = \begin{bmatrix} -a & a \\ c - a - z & c \end{bmatrix}.$$

Considering current knowledge on fractional order systems, it is difficult, if not impossible, to analyze projective synchronization in fractional order systems theoretically. We numerically study this topic here.

We consider a coupled Chen system with the fractional order $\alpha = 0.9$ in Eq. (3). We let the parameters $(a, b, c) = (35, 3, 28)$ in Eq. (5), so that the fractional order Chen system is chaotic [17]. The initial values for the state variables are some random values close to the origin. The two chaotic oscillators can always achieve projective synchronization in our simulations by using the method mentioned above. The dynamical behaviors of the master and slave systems in a simulation are shown in Fig. 1. In this figure, the initial values of the coupled system are $(x_m, y_m, z, x_s, y_s) = (0.0440, 0.0701, 0.0610, 0.0300, 0.0856)$. In Fig. 1 (a), we show the time evolution of the scaling factor x_s/x_m , which indicates that the scaling factor converges to a constant value. In Fig. 1 (b), we show the projections of the master system and the slave system onto the $x - y$ plane. As we see that the two attractors are the same in structure but different in size, which also clearly indicates the projective synchronization of the coupled fractional order system.

Because the scaling factor in projective synchronization is dependent on the initial condition and unpredictable, in [29, 30] the authors proposed some control methods for controlling the scaling factor β onto a desired value β^* . In this paper, we also present a simple feedback control mechanism for the coupled fractional order system. We introduce a controller to the slave system, and the coupled system now becomes

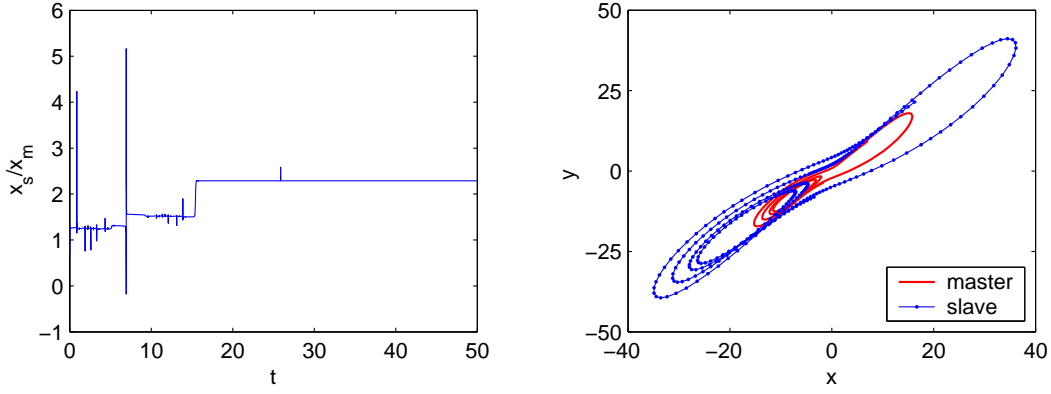


Figure 1: (a) Time evolution of the scaling factor x_s/x_m . (b) The projection of the master and slave system onto the $x - y$ plane.

$$\begin{aligned}
 \frac{d^\alpha \mathbf{u}_m}{dt^\alpha} &= \mathbf{M}(z) \cdot \mathbf{u}_m, \\
 \frac{d^\alpha z}{dt^\alpha} &= f(\mathbf{u}_m, z), \\
 \frac{d^\alpha \mathbf{u}_s}{dt^\alpha} &= \mathbf{M}(z) \cdot \mathbf{u}_s + k(\beta^* \mathbf{u}_m - \mathbf{u}_s)
 \end{aligned} \tag{6}$$

where k is a positive constant and β^* is the desired scaling factor. The control mechanism is standard in integer-order systems (see e.g. [31]), and can be easily understood. If any component of \mathbf{u}_s is larger than the corresponding component of $\beta^* \mathbf{u}_m$, then the corresponding component of $k(\beta^* \mathbf{u}_m - \mathbf{u}_s)$ will be negative and the rate of that component of \mathbf{u}_s will be decreased, thus $\mathbf{u}_s/\mathbf{u}_m$ will also be decreased, and vice versa. So eventually the scaling factor should asymptotically converge to β^* .

We let $k = 0.1$, and apply the control mechanism to the coupled system. The desired scaling factor β^* can be reached for any reasonable positive and negative values with any random initial conditions. In Fig. 2 (a) and (b), we show the projections of the master and slave systems onto the $x - y$ plane for $\beta^* = 5$ and -3 , respectively. As we see that the two attractors in each panel of this figure are the same in structure but different in size (and direction). The size of the master system does not change, but the slave system is amplified. By numerous simulations, we found that, even for a very small k (say $k = 0.01$), after a long time, the two oscillators can also be synchronized up to the constant scaling factor β^* .

For other fractional orders and some other parameter values, we also found similar phenomena. We omit these results here.

In summary, we have studied the projective synchronization in coupled two fractional order chaotic oscillators. It is shown that projective synchronization can also exist in coupled fractional order chaotic systems. A simple feedback control method is also presented in this paper, which can drive the scaling factor onto a desired value. Theoretical analysis of various synchronization phenomena, including projective synchronization, in fractional order chaotic systems will be the subject of future research.

This work was supported by the National Natural Science Foundation of China under Grant 60502009, and the Program for New Century Excellent Talents in University of China.

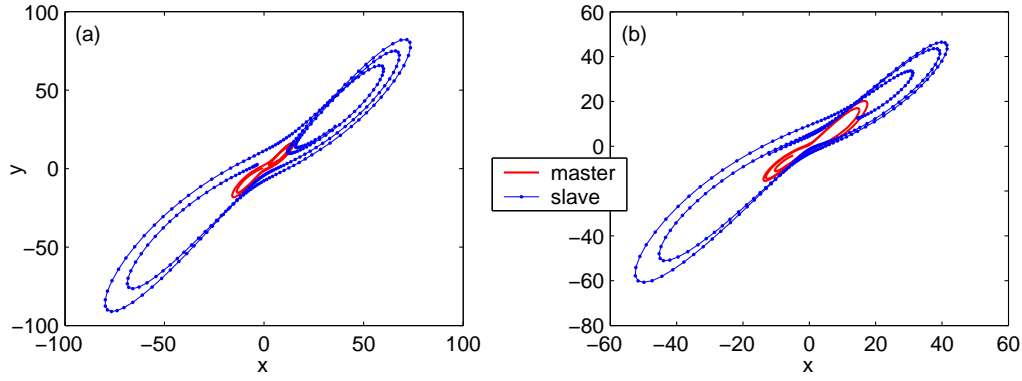


Figure 2: The projections of the master and slave systems onto the $x - y$ plane for (a) $\beta^* = 5$; (b) $\beta^* = -3$.

References

- [1] I. Podlubny, *Fractional Differential Equations*, (Academic Press, New York, 1999).
- [2] R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics*, (World Scientific, New Jersey, 2001).
- [3] R.L. Bagley, and R.A. Calico, J. Guid., Contr. Dyn. 14, 304 (1991).
- [4] R. C. Koeller, J. Appl. Mech. 51, 299 (1984).
- [5] R.C. Koeller, Acta Mechanica 58, 251 (1986).
- [6] H.H. Sun, A.A. Abdelwahad, and B. Onaral, IEEE Trans. Auto. Contr. 29, 441 (1984).
- [7] M. Ichise, Y. Nagayanagi, and T. Kojima, J. Electroanal. Chem. 33, 253 (1971).
- [8] O. Heaviside, *Electromagnetic Theory*, (Chelsea, New York, 1971).
- [9] T.T. Hartley, C.F. Lorenzo, and H. K. Qammer, IEEE Trans. CAS-I 42, 485 (1995).
- [10] P. Arena, R. Caponetto, L. Fortuna, and D. Porto, Proc. ECCTD, Budapest, 1259 (1997).
- [11] W.M. Ahmad, and J.C. Sprott, Chaos, Solitons and Fractals 16, 339 (2003).
- [12] W.M. Ahmad, and W.M. Harb, Chaos, Solitons and Fractals 18, 693 (2003).
- [13] I. Grigorenko, and E. Grigorenko, Phys. Rev. Lett. 91, 034101 (2003). But the Eq. (5) in this paper is not correct, so the results presented in this paper are not reliable, which I have pointed out to the first author of this paper via personal communication.
- [14] P. Arena, R. Caponetto, L. Fortuna, and D. Porto, Int. J. Bifur. Chaos 7, 1527 (1998).
- [15] P. Arena, L. Fortuna, and D. Porto, Phys. Rev. E 61, 776 (2000).
- [16] C. Li, G. Chen, Physica A 341, 55 (2004).
- [17] C. Li, G. Chen, Chaos, Solitons and Fractals 22, 549 (2004).

- [18] G.M. Zaslavsky, Phys. Rep. 371, 461 (2002).
- [19] S. Boccaletti, J. Kurths, G. Osipov, D.L. Valladares, and C.S. Zhou, Phys. Rep. 366, 1 (2002).
- [20] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. 64, 821 (1990).
- [21] M. Rosenblum, A. Pikovsky, and J. Kurtz, Phys. Rev. Lett. 76, 1804 (1996);
A. Pikovsky, M. Rosenblum, G. Osipov, and J. Kurtz, Physica D 104, 219 (1997).
- [22] M. G. Rosenblum, A.S. Pikovsky, J. Kurths, Phys. Rev. Lett. 78, 4193 (1997).
- [23] R. Mainieri and J. Rehacek, Phys. Rev. Lett. 82, 3042 (1999).
- [24] C. Li, X. Liao, J. Yu, Phys. Rev. E 68, 067203 (2003).
- [25] W.H. Deng, C.P. Li, J. Phys. Soc. Jpn 74, 1045, 2005;
J. G. Lu, Chaos, Solitons and Fractals 26, 1125, 2005;
X. Gao, Chaos, Solitons and Fractals 26, 141, 2005;
J. G. Lu, Chaos, Solitons and Fractals 27, 519, 2006;
H. Zhang, C.G. Li, G. Chen, Int. J. Mod. Phys. C 16, 815, 2005;
W.H. Deng, C.P. Li, Physica A 353, 61, 2005.
- [26] C. Li, Phase and lag synchronization in coupled fractional order chaotic oscillators, IEEE Trans. Circuits and Systems-II, submitted.
- [27] M. Moshrefi-Torbati, J.K. Hammond, J. Franklin Inst. 335B, 1077 (1998);
I. Podlubny, arXiv: math.CA/0110241 (2001).
- [28] A. Charef, H.H. Sun, Y.Y. Tsao, and B. Onaral, IEEE Trans. Auto. Contr. 37, 1465 (1992).
- [29] D. Xu, Phys. Rev. E 63, 027201 (2001).
- [30] D. Xu, Z. Li, Int. J. Bifur. Chaos 12, 1395 (2002).
- [31] J C. Doyle, B. A. Francis, A. R. Tannenbaum, *Feedback Control Theory* (Macmillan Coll Div, 1992).